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*Full Length Research Paper*

# Mass transfer analysis for unsteady thin film flow over stretched heated plate

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**In this paper, the mass transfer in a time varying thin liquid film over a stretching heated plate having variable temperature and concentration is analyzed. The analytical solution for the unsteady Navier-Stokes, energy and diffusion equations are obtained. A new similarity is found using group-theoretic analysis which renders the exact similar governing equations amenable to analytical solution using perturbation method. Numerical solution of the problem is also obtained showing good agreement with analytical results. Graphs of velocity, temperature, concentration, skin-friction, Nusselt number and Sherwood number are displayed for various values of pertinent parameters.**

**Key words:** Thin film, mass transfer, group-theoretic analysis, perturbation solution.

## INTRODUCTION

Thin films have many applications in industry, lubrications of machinery, fluid bearings, coating (Andersson et al., 2000) including the preparation of thin films, printing and painting and in the adhesives. Biological applications include studies of liquid flow in the lungs and eyes. Some further areas of applications are wire and fiber coating, foodstuff processing, reactor fluidization, transpiration, cooling, polymer processing etc (Abel et al., 2009).

Crane (1970) was the first to examine the semi-infinite fluid flow driven by a linearly stretching surface. Later on, Gupta and Gupta (1977), Vleggaar (1977), Carragher and Crane (1982), Grubka and Bobba (1985), Dutta et al. (1985), Jeng et al. (1986), Chen and Char (1988), Kumari et al. (1990) and Dandapat et al. (2004) studied various aspects of this problem, such as heat, mass and momentum transfer in viscous flows with or without suction through the sheet. All these studies were however made to study steady flow in a semi-infinite fluid layer driven by a continuous stretching sheet. The hydrodynamics of thin liquid film on an unsteady stretching sheet was first considered by Wang (1990). The time dependent Navier-Stokes equation is reduced to ordinary differential equation (ODE) through similarity transformations with an unsteady parameter. Later on, Usha and Sridharan (1993) considered the similar problem of axisymmetric flow in a liquid film. Andersson et al. (2000) extended Wang's problem by analyzing heat transfer in the liquid film driven by an unsteady stretching surface. Wang's work was further extended to include the non-Newtonian effects of fluid, heat transfer and the thermocapillarity effects by Dandapat et al. (2003). They discussed the physical mechanisms that govern the thermal characteristics for various Prandtl numbers and different values of the unsteadiness parameter *S* and presented the numerical solution after using a similarity transformation. The similarity transformations used by Andersson et al., (2000) transform the governing equations into a locally similar problem for which the numerical solution is presented. Later on, Chun Wang (2006) gave the homotopy analysis method (HAM) solution for the same problem and giving similarity transformation.

Applications of thin film in heat exchangers, chemical processes, wire coating and in food processing also require understanding of mass transfer by the flow of thin film on a stretching sheet. According to the author's knowledge, there are very few or no studies on mass transfer by a thin film over a stretched sheet, so in

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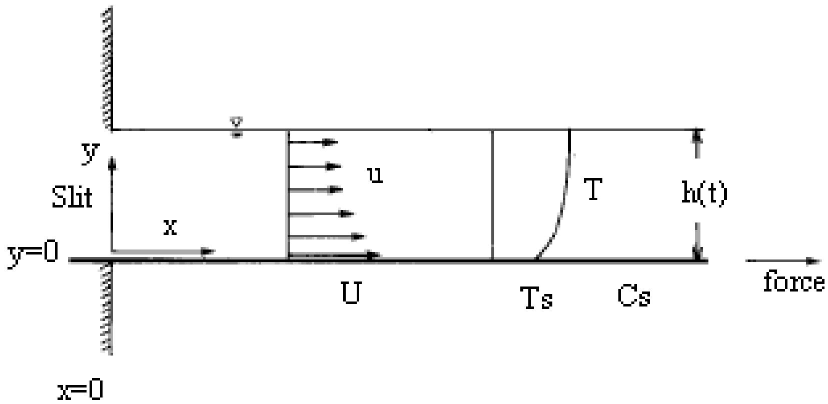


**Figure 1.** Schematic diagram of the problem.

addition to giving new similarity for the thin film flow, we also present here the mass transfer analysis by a thin film over a stretched heated plate. We introduce a new similarity using group-theoretic method which reduces the problem into self similar with a single parameter appearing in the equations. This parameter appears at a very convenient position suitable for perturbation expansion. Regular perturbation method is applied to get analytical solution. Numerical solution of the problem is also obtained and a very good agreement is found with analytical solution.

## FORMULATION OF THE PROBLEM

A uniform thin film of viscous fluid of height $h(t)=\sqrt{\upsilon(1-ct)/b}$ covers a sheet along *x*-axis where the sheet is emerged from a slit and is being stretched linearly with a velocity given by:

$$U=-\frac{\dot{h}(t)x}{h(t)}. \tag{1}$$

Here, it is pertinent to note that we have used general stretching velocity. In the expression of *h(t)* stated earlier, both *b* and *c* are positive constants having dimension $\mathrm{time}^{-1}$ and $\upsilon$ is the kinematic viscosity. This formulation of the height $h(t)$ is valid only for $ct<1$. The temperature and concentration at the surface of the sheet ($T_s$ and $C_s$) are specified as:

$$T_s=T_0-T_{ref}\,g\left(x,t\right),\; C_s=C_0-C_{ref}\,m\left(x,t\right), \tag{2}$$

where $T_0$ and $C_0$ represent the slit temperature and concentration. In Equation 2, any constant temperature and concentration can be taken as reference temperature $T_{ref}$ and reference concentration $C_{ref}$. In Equation 2, $g\left(x,t\right)$ and $m\left(x,t\right)$ are arbitrary functions to be determined by self similar condition. Flow geometry and the coordinate system are shown in the Figure 1.

The governing unsteady equations which describe the motion of the viscous fluid over the stretching sheet are the continuity equation, Navier-Stokes (boundary layer) equations, energy equation and the concentration equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2} \tag{4}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{5}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2} \tag{6}$$

with following boundary conditions

$$\begin{array}{lllll} u=U, & v=0, & T=T_s, & C=C_s & \text{at} \quad y=0 \\ \dfrac{\partial u}{\partial y}=0, & v=A\dfrac{dh}{dt}, & \dfrac{\partial T}{\partial y}=0, & \dfrac{\partial C}{\partial y}=0 & \text{at} \quad y=h(t), \end{array} \tag{7}$$

where *A* is non-dimensional adjusting parameter, $\alpha$ is the thermal diffusivity and *D* is the molecular diffusivity of the mass transport. The free surface condition at the surface of thin film requires tangential stress to be negligible. Thus, boundary condition $\partial u/\partial y=0$ at $y=h(t)$ tacitly assume that the film height and the boundary layer merges with each other. The surface of liquid film is assumed free of surface waves and smooth. Hence, due to inert atmosphere, interfacial shear and surface tension is neglected as it is done by other researchers for example, Andersson (2000). In this study, the cross diffusion effects are also assumed to be negligible compared with direct effects, modeled by Fourier's law and Fick's law as suggested by Gebhart and Pera (1971a, 1971b).

**SIMILARITY ANALYSIS**

We now develop similarity relations to reduce the equations into ODEs which would then be solved conveniently using both analytical and numerical method. We make a little digression from the commonly used similarity transformation which in fact generates non-similar problem. The new similarity transformation will be built using group-theoretic approach. The physical reason being that it is advisable to non-dimensionalize the velocity by the velocity of the deforming thin layer rather by the free stream velocity. The scaling transformation is found to be the best approach to achieve our end.

Firstly, let us introduce the following non-dimensional variables

$$\tilde{x}=\frac{x}{L},\ \tilde{y}=\frac{y}{L},\ \tilde{u}=\frac{u}{U_0},\ \tilde{v}=\frac{v}{U_0},\ \tilde{t}=t\frac{U_0}{L},\ \tilde{T}=\frac{T-T_0}{T_s-T_0},\tilde{C}=\frac{C-C_0}{C_s-C_0},\tilde{h}=\frac{h}{L}. \quad (8)$$

Under the new variables, the governing equations (3) to (6) take the form

$$\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\partial\tilde{v}}{\partial\tilde{y}}=0 \quad (9)$$

$$\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=\frac{1}{\mathrm{Re}}\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2} \quad (10)$$

$$\frac{\partial\tilde{T}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{T}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{T}}{\partial\tilde{y}}+\tilde{T}\left(\frac{1}{g(\tilde{x},\tilde{t})}\frac{\partial g}{\partial\tilde{t}}+\frac{\tilde{u}}{g(\tilde{x},\tilde{t})}\frac{\partial g}{\partial\tilde{x}}\right)=\frac{1}{\mathrm{Pr.Re}}\frac{\partial^2\tilde{T}}{\partial\tilde{y}^2} \quad (11)$$

$$\frac{\partial\tilde{C}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{C}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{C}}{\partial\tilde{y}}+\tilde{C}\left(\frac{1}{m(\tilde{x},\tilde{t})}\frac{\partial m}{\partial\tilde{t}}+\frac{\tilde{u}}{m(\tilde{x},\tilde{t})}\frac{\partial m}{\partial\tilde{x}}\right)=\frac{1}{\mathrm{Sc.Re}}\frac{\partial^2\tilde{C}}{\partial\tilde{y}^2} \quad (12)$$

with the corresponding boundary conditions

$$\begin{aligned}&\tilde{u}=\tilde{U},\ \tilde{v}=0, \quad \tilde{T}=\tilde{T}_s, \quad \tilde{C}=\tilde{C}_s, \quad \text{at} \quad \tilde{y}=0\\&\frac{\partial\tilde{u}}{\partial\tilde{y}}=0,\ \tilde{v}=A\frac{d\tilde{h}}{d\tilde{t}}, \quad \frac{\partial\tilde{T}}{\partial\tilde{y}}=0, \quad \frac{\partial\tilde{C}}{\partial\tilde{y}}=0, \quad \text{at} \quad \tilde{y}=\tilde{h}(t)\end{aligned} \quad (13)$$

From Equations 9 to 13, $\mathrm{Sc}=\upsilon/D$ is Schmidt number, $\mathrm{Pr}=\upsilon/\alpha$ is Prandtl number and $\mathrm{Re}=\mathrm{UL}/\upsilon$ is Reynolds number. To make the equations parameter free, we take

$$\hat{x}=\tilde{x},\ \hat{y}=\tilde{y}\sqrt{\mathrm{Re}},\ \hat{t}=\tilde{t},\ \hat{u}=\tilde{u},\ \hat{v}=\tilde{v}\sqrt{\mathrm{Re}},\ \hat{T}=\tilde{T},\ \hat{C}=\tilde{C},\ \hat{h}=\tilde{h}\sqrt{\mathrm{Re}}. \quad (14)$$

The Equations 9 to 13 thus, transform into:

$$\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}=0 \quad (15)$$

$$\frac{\partial\hat{u}}{\partial\hat{t}}+\hat{u}\frac{\partial\hat{u}}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{u}}{\partial\hat{y}}=\frac{\partial^2\hat{u}}{\partial\hat{y}^2} \quad (16)$$

$$\frac{\partial\hat{T}}{\partial\hat{t}}+\hat{u}\frac{\partial\hat{T}}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{T}}{\partial\hat{y}}+\hat{T}\left(\frac{1}{g(\hat{x},\hat{t})}\frac{\partial g}{\partial\hat{t}}+\frac{\hat{u}}{g(\hat{x},\hat{t})}\frac{\partial g}{\partial\hat{x}}\right)=\frac{1}{\mathrm{Pr}}\frac{\partial^2\hat{T}}{\partial\hat{y}^2} \quad (17)$$

$$\frac{\partial\hat{C}}{\partial\hat{t}}+\hat{u}\frac{\partial\hat{C}}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{C}}{\partial\hat{y}}+\hat{C}\left(\frac{1}{m(\hat{x},\hat{t})}\frac{\partial m}{\partial\hat{t}}+\frac{\hat{u}}{m(\hat{x},\hat{t})}\frac{\partial m}{\partial\hat{x}}\right)=\frac{1}{\mathrm{Sc}}\frac{\partial^2\hat{C}}{\partial\hat{y}^2} \quad (18)$$

with the boundary conditions

$$\begin{aligned}&\hat{u}=\hat{U},\ \hat{v}=0, \quad \hat{T}=\hat{T}_s, \quad \hat{C}=\hat{C}_s, \quad \text{at} \quad \hat{y}=0\\&\frac{\partial\hat{u}}{\partial\hat{y}}=0,\ \hat{v}=A\frac{d\hat{h}}{d\hat{t}}, \quad \frac{\partial\hat{T}}{\partial\hat{y}}=0, \quad \frac{\partial\hat{C}}{\partial\hat{y}}=0, \quad \text{at} \quad \hat{y}=\hat{h}\end{aligned} \quad (19)$$

The scaling transformation suggests to write

$$\begin{aligned}&x^*=e^{\varepsilon a}\hat{x},\ \ y^*=e^{\varepsilon b}\hat{y},\ t^*=e^{\varepsilon c}\hat{t},\ \ u^*=e^{\varepsilon p}\hat{u},\ v^*=e^{\varepsilon q}\hat{v},\ h^*=e^{\varepsilon i}\hat{h},\ U^*=e^{\varepsilon j}\hat{U},\\&T^*=e^{\varepsilon k}\hat{T},\ C^*=e^{\varepsilon l}\hat{C},\ g^*=e^{\varepsilon n}\hat{g},\ m^*=e^{\varepsilon o}\hat{m}.\end{aligned} \quad (20)$$

Here, $\varepsilon$ is transformation parameter and $a,b,c,p,q,i,j,k,l,n,o$ are arbitrary parameters to be determined by the invariance condition on the equations. Equations 15 to 19 take the form:

$$\frac{\partial u^*}{\partial x^*}+e^{\varepsilon(-a+p+b-q)}\frac{\partial v^*}{\partial y^*}=0 \quad (21)$$

$$\frac{\partial u^*}{\partial t^*}+e^{\varepsilon(a-c-p)}u^*\frac{\partial u^*}{\partial x^*}+e^{\varepsilon(b-c-q)}v^*\frac{\partial u^*}{\partial y^*}=e^{\varepsilon(2b-c)}\frac{\partial^2u^*}{\partial y^{*2}} \quad (22)$$

$$\begin{aligned}&\frac{\partial T^*}{\partial t^*}+e^{\varepsilon(a-c-p)}u^*\frac{\partial T^*}{\partial x^*}+e^{\varepsilon(b-c-q)}v^*\frac{\partial T^*}{\partial y^*}+T^*\frac{1}{g^*(x^*,t^*)}\frac{\partial g^*}{\partial t^*}\\&+e^{\varepsilon(a-c-p)}T^*\frac{u^*}{g^*(x^*,t^*)}\frac{\partial g^*}{\partial x^*}=\frac{1}{\mathrm{Pr}}e^{\varepsilon(2b-c)}\frac{\partial^2T^*}{\partial y^{*2}}\end{aligned} \quad (23)$$

$$\begin{aligned}&\frac{\partial C^*}{\partial t^*}+e^{\varepsilon(a-c-p)}u^*\frac{\partial C^*}{\partial x^*}+e^{\varepsilon(b-c-q)}v^*\frac{\partial C^*}{\partial y^*}+C^*\frac{1}{m^*(x^*,t^*)}\frac{\partial m^*}{\partial t^*}\\&+e^{\varepsilon(a-c-p)}C^*\frac{u^*}{m^*(x^*,t^*)}\frac{\partial m^*}{\partial x^*}=\frac{1}{\mathrm{Sc}}e^{\varepsilon(2b-c)}\frac{\partial^2C^*}{\partial y^{*2}}\end{aligned} \quad (24)$$

$$\begin{aligned}&u^*=e^{\varepsilon(p-j)}U^*,\ \ v^*=0,\ \ T^*=e^{\varepsilon(-k)}1, \quad C^*=e^{\varepsilon(-l)}1 \quad \text{at} \quad y^*=0\\&\frac{\partial u^*}{\partial y^*}=0,\ \ v^*=e^{\varepsilon(q+c-i)}\frac{dh^*}{dt^*},\ \ \frac{\partial T^*}{\partial y^*}=0,\ \frac{\partial C^*}{\partial y^*}=0 \quad \text{at} \quad y^*=h^*\end{aligned} \quad (25)$$

The similarity will be achieved by imposing the conditions

$$-a+p+b-q=0,\ a-c-p=0,\ b-c-q=0,\ \ 2b-c=0,\ \ p-j=0,\ q+c-i=0 \quad (26)$$

The system of equations can be solved for the constants *b* and *d* giving

$$a=(2+p_0)b,\ c=2b,\ p=p_0b,\ q=-b,\ i=b,\ \ j=p_0b, k=0, l=0 \quad (27)$$

The corresponding equivalent differential system is

$$\frac{d\hat{x}}{a\hat{x}}=\frac{d\hat{y}}{b\hat{y}}=\frac{d\hat{t}}{c\hat{t}}=\frac{d\hat{u}}{d\hat{u}}=\frac{d\hat{v}}{e\hat{v}}=\frac{d\hat{h}}{i\hat{h}}=\frac{d\hat{U}}{j\hat{U}}=\frac{d\hat{T}}{k\hat{T}}=\frac{d\hat{C}}{l\hat{C}} \quad (28)$$

Choosing the value for the parameter $p=p_0b$, where the $p_0$ is another parameter, Equation 28 becomes

$$\frac{d\hat{x}}{(p_0+2)\hat{x}}=\frac{d\hat{y}}{\hat{y}}=\frac{d\hat{t}}{2\hat{t}}=\frac{d\hat{u}}{p_0\hat{u}}=\frac{d\hat{v}}{-\hat{v}}=\frac{d\hat{h}}{\hat{h}}=\frac{d\hat{U}}{p_0\hat{U}}=\frac{d\hat{T}}{0}=\frac{d\hat{C}}{0}. \quad (29)$$

The similarity variable and the similarity functions are now obtained as

$$\eta = \frac{\hat{y}}{\hat{h}},\ \hat{T} = \theta(\eta),\ \hat{C} = \phi(\eta),\ \hat{u} = \hat{U} f'(\eta), \tag{30}$$

Giving

$$\eta = \frac{y}{h(t)},\ T = T_0 - T_{ref}\, g(x,t)\,\theta(\eta),$$
$$C = C_0 - C_{ref}\, m(x,t)\,\phi(\eta),\ u = -\frac{\dot{h}(t)x}{h(t)} f'(\eta), \tag{31}$$

through Equation 8, 14 and 20. By using the similarity function of Equation 30, the stream function can be expressed as:

$$\psi = -\dot{h} x f(\eta). \tag{32}$$

The *y* component of velocity takes the form

$$v = -\frac{\partial \psi}{\partial x} = \dot{h}(t)\, f(\eta). \tag{33}$$

With the help of Equations 26 and 27, Equations 3 to 6 become

$$f''' - \frac{h^2 \ddot{h}}{\upsilon \dot{h}} f' + \frac{h\dot{h}}{\upsilon}[f' + \eta f'' - (f')^2 + f f''] = 0, \tag{34}$$

$$\theta'' + \Pr\left(\frac{h\dot{h}}{\upsilon}\right)\left[\left\{\frac{x}{g(x,t)}\frac{\partial g(x,t)}{\partial x}\right\}\theta + \eta\theta' + \left\{\frac{1}{g(x,t)}\frac{\partial g(x,t)}{\partial t}\left(\frac{h}{\dot{h}}\right)\right\}\theta f' + \theta' f\right] = 0. \tag{35}$$

$$\phi'' + \mathrm{Sc}\left(\frac{h\dot{h}}{\upsilon}\right)\left[\left\{\frac{x}{m(x,t)}\frac{\partial m(x,t)}{\partial x}\right\}\phi + \eta\phi' + \left\{\frac{1}{m(x,t)}\frac{\partial m(x,t)}{\partial t}\left(\frac{h}{\dot{h}}\right)\right\}\phi f' + \phi' f\right] = 0. \tag{36}$$

We have been successful in reaching out ODEs from partial differential equations but short of achieving complete similarity since the terms in Equations 34 to 36 do contain the old variables *x* and *t*. In order to achieve a complete symmetry in Equations 34 to 36, we emphasize

$$g(x,t) = \left(\frac{x}{h}\right)^r,\ m(x,t) = \left(\frac{x}{h}\right)^s,\ \ \gamma = -\frac{h\dot{h}}{\upsilon} = \text{constant}$$

where *r* and *s* are non negative real numbers. With this choice of $g(x,t)\,\text{and}\, m(x,t)$ Equations 34 to 36 finally take the form

$$f''' - \gamma\left[2f' + \eta f'' + (f')^2 - f f''\right] = 0, \tag{37}$$

$$\theta'' - \Pr\gamma\left[r\theta + \eta\theta' + r\theta f' - \theta' f\right] = 0, \tag{38}$$

$$\phi'' - \mathrm{Sc}\gamma\left[s\phi + \eta\phi' + s\phi f' - \phi' f\right] = 0, \tag{39}$$

satisfying the boundary conditions

$$\begin{aligned} &f'(0) = 1, \quad f(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1, \\ &f''(1) = 0, \quad f(1) = A, \quad \theta'(1) = 0, \quad \phi'(1) = 0. \end{aligned} \tag{40}$$

The significance of the proposed similarity transformation is quite apparent from equations (37)-(39) in which not only self similarity has been obtained but also a parameter occupies a very suitable place for perturbation analysis.

## ANALYTICAL SOLUTION

In this section, we solve the problem given by Equations 37 to 39 with boundary conditions of Equation 40. It is quite reasonable to assume $\gamma$ as a small parameter which is true for many engineering applications.

For small alpha, we construct a straight forward expansion of the form

$$f = \sum_{n=0}^{\infty}\gamma^n f_n,\ \theta = \sum_{n=0}^{\infty}\gamma^n \theta_n \text{ and } \phi = \sum_{n=0}^{\infty}\gamma^n \phi_n\ . \tag{41}$$

It is advisable to write *A* in a series in alpha such that

$$A = \sum_{n=0}^{\infty}\gamma^n A_n, \tag{42}$$

where $A_n$'s are to be determined.

The unperturbed leading order system, from equations (37)-(40) is given by

$$f_0''' = 0, \tag{43a}$$

$$\theta_0'' = 0, \tag{43b}$$

and $$\phi_0'' = 0, \tag{43c}$$

with the boundary conditions

$$\begin{aligned} &f_0'(0) = 1, \quad f_0(0) = 0, \quad \theta_0(0) = 1, \quad \theta_0(0) = 1, \\ &f_0''(1) = 0, \quad f_0(1) = A_0, \quad \theta_0'(1) = 0, \quad \phi_0'(1) = 0. \end{aligned} \tag{44}$$

The kth order systems for $k \geq 1$ can be expressed as:

$$f_k''' - 2f_{k-1}' - \eta f_{k-1}'' - \sum_{i=1}^{k}\left(f_{k-i}' f_{i-1}' + f_{k-i} f_{i-1}''\right) = 0, \tag{45a}$$

$$\theta_k'' + \Pr\left[-r\theta_{k-1} - \eta\theta_{k-1}' - r\sum_{i=1}^{k}\left(\theta_{k-i} f_{i-1}' + \theta_{k-i}' f_{i-1}\right)\right] = 0, \tag{45b}$$

$$\phi_k'' + \mathrm{Sc}\left[-s\phi_{k-1} - \eta\phi_{k-1}' - \sum_{i=1}^{k}\left(s\phi_{k-i} f_{i-1}' + \phi_{k-i}' f_{i-1}\right)\right] = 0, \tag{45c}$$

$$f_k'(0)=0,\quad f_k(0)=0,\quad \theta_k(0)=0,\quad \phi_k(0)=0,$$
$$f_k''(1)=0,\quad f_k(1)=A_k,\quad \theta_k'(1)=0,\quad \phi_k'(1)=0. \tag{46}$$

The leading order of Equations 43 with boundary conditions of Equation 44 has the solution

$$f_0=\eta,\ \theta_0=1,\ \phi_0=1. \tag{47}$$

Four term perturbation solution that is, $O(\gamma^3)$ for $f$, $\theta$ and $\phi$ can finally be written as in the Equations 48 to 50 and $A_k$'s turn out to be one.

$$f=\eta+\frac{\gamma}{2}(-3\eta^2+\eta^3)+\frac{\gamma^2}{10}(20\eta^2-5\eta^4+\eta^5)+\frac{\gamma^3}{840}(-3066\eta^2+560\eta^4+63\eta^5-77\eta^6+11\eta^7), \tag{48}$$

$$\begin{aligned}\theta&=1-2\gamma r\Pr(\eta-\eta^2)+\frac{\gamma^2}{24}\{r\Pr(24\eta-12\eta^3+3\eta^4)+r^2\Pr^2(32\eta-16\eta^3+4\eta^4)\}\\&+\frac{\gamma^3}{360}\{(r\Pr(-576\eta+240\eta^3-36\eta^5+6\eta^6)+r\Pr^2(72\eta-90\eta^4+72\eta^5-12\eta^6)\\&+r^2\Pr^2(-576\eta+120\eta^3+180\eta^4-126\eta^5+21\eta^6)+r^3\Pr^3(-384\eta+160\eta^3-24\eta^5+4\eta^6))\},\end{aligned} \tag{49}$$

$$\begin{aligned}\phi&=1-2\gamma s\mathrm{Sc}(\eta-\eta^2)+\frac{\gamma^2}{24}\{s\mathrm{Sc}(24\eta-12\eta^3+3\eta^4)+s^2\mathrm{Sc}^2(32\eta-16\eta^3+4\eta^4)\}\\&+\frac{\gamma^3}{360}\{(s\mathrm{Sc}(-576\eta+240\eta^3-36\eta^5+6\eta^6)+s\mathrm{Sc}^2(72\eta-90\eta^4+72\eta^5-12\eta^6)\\&+s^2\mathrm{Sc}^2(-576\eta+120\eta^3+180\eta^4-126\eta^5+21\eta^6)+s^3\mathrm{Sc}^3(-384\eta+160\eta^3-24\eta^5+4\eta^6))\}.\end{aligned} \tag{50}$$

## NUMERICAL SOLUTION

The numerical solution of the boundary value problem is obtained using shooting method in conjunction with sixth-order Runge-Kutta integration. For this purpose, we rewrite Equations 37 to 40 as:

$$f'''=\gamma\left[2f'+\eta f''+(f')^2-ff''\right], \tag{51}$$

$$\theta''=\Pr\gamma\left[r\theta+\eta\theta'+r\theta f'-\theta' f\right] \tag{52}$$

$$\phi''=\mathrm{Sc}\gamma\left[s\phi+\eta\phi'+s\phi f'-\phi' f\right], \tag{53}$$

with the boundary conditions

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=g_1$$
$$\theta(0)=1,\quad \theta'(0)=g_2,\quad \phi(0)=1,\quad \phi'(0)=g_3. \tag{54}$$

where $g_1$, $g_2$ and $g_3$ are the missing initial conditions and can be determined by employing shooting method that is linked with sixth-order Runge-Kutta method to find the solution of problem satisfying the given boundary conditions.

Having the solution of the problem, we focus our attention to the other important quantities from the technological point of view that are the $\tau_w$ shear stress, rate of heat transfer at the surface $q_w$ and rate of mass transfer at the surface $m_w$, defined as:

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0},\quad q_w=-k\left(\frac{\partial T}{\partial y}\right)_{y=0},\quad m_w=-D\left(\frac{\partial C}{\partial y}\right)_{y=0} \tag{55}$$

The physical quantities can be written in terms of skin friction $C_f$, Nusselt number, Nu and Sherwood number, Sh, defined by:

$$C_f=\frac{\tau_w}{\rho U^2},\quad \mathrm{Nu}=\frac{xq_w}{k\Delta T},\quad \mathrm{Sh}=\frac{xm_w}{D\Delta C} \tag{56}$$

Here, $\Delta T=T_s-T_0$ and $\Delta C=C_s-C_0$ are constant temperature and concentration differences. Using non-dimensional variables given in Equation 31, the skin friction, $C_f$, the Nusselt number, Nu and the Sherwood number, Sh, become:

$$C_f=-\frac{\upsilon f''(0)}{xh'}\ \text{or}\ -\mathrm{Re}_x^{-1}f''(0),\quad \mathrm{Nu}=-x\theta'(0)/h,\quad \mathrm{Sh}=-x\phi'(0)/h. \tag{57}$$

Here, $\mathrm{Re}_x=h'x/\upsilon$ is the local Reynolds number based on the velocity of thin film boundary.

## RESULTS AND DISCUSSION

The graphs of velocity, temperature and concentration profiles as shown, respectively in the Figures 2(a), (b) and (c), exhibit that all these quantities decrease with the increase in $\gamma$ deformation ratio of the boundary of thin film. The variation of temperature with the Pr and r is shown in Figure 3. Figure 3 shows that temperature is decreased with increase of both of these parameters. This is due to the reason that increase of Pr decreases the conductivity of the fluid so temperature decreases. Same effect is observed on concentration in Figure 4 due to increase of Schmidt number Sc and parameter s, as here increase in Sc decreases the molecular diffusivity *D* which in turn reduces the concentration.

The effect of important parameters on skin-friction, mass transfer rate and on heat transfer rate is shown in Figures 5 to 7. The Figures 5 to 7 also support the behaviors of velocity, concentration and temperature already shown in Figures 2 to 4. As large values of Prandtl number, Pr, decrease the temperature which gives rise to a larger temperature gradient and ultimately heat transfer rate increases. Due to the similar reason, mass transfer rate also increases with increase in Schmidt number Sc.

## CONCLUSIONS

The mass transfer in a time varying thin liquid film over a stretching heated plate was obtained. The following objectives were achieved. A new similarity was established using group-theoretic analysis. A self similar analytical solution was obtained using this similarity which is in excellant agreement for a wide range of Prandtl and Schmidt number but for small values of deformation ratio of thin film boundary $\gamma$ as it was used as perturbation parameter. However, numerical solution was also obtained and can be used for wide range of all the parameters.

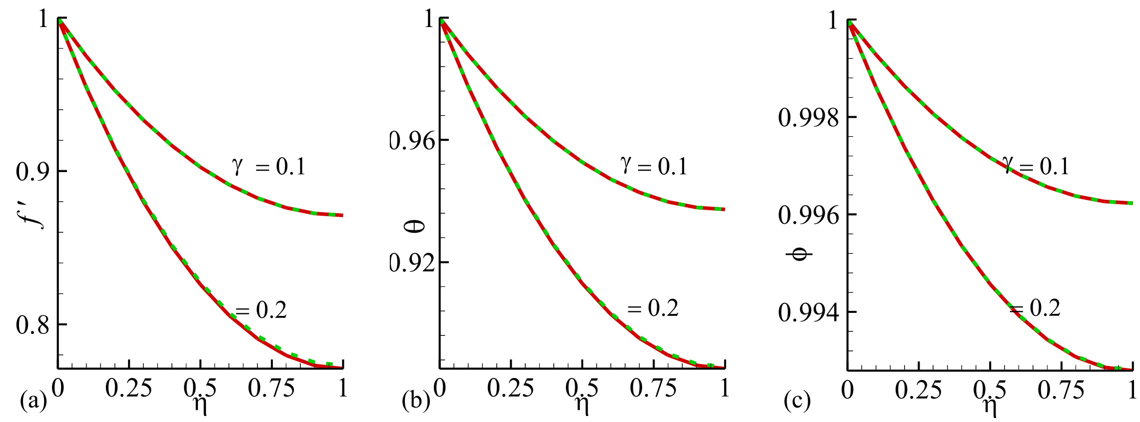


**Figure 2.** (a), Velocity; (b), temperature; (c), concentration profiles for different values of $\gamma$ while Sc=0.2, s=0.2, Pr=0.7 and r=1.0.

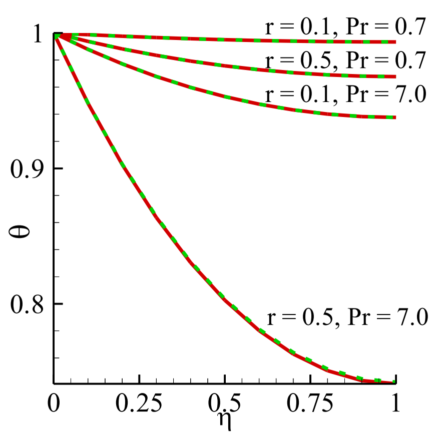


**Figure 3.** Temperature profiles for various values of r and Pr while $\gamma$ =0.1, s=0.1, Sc=0.1.

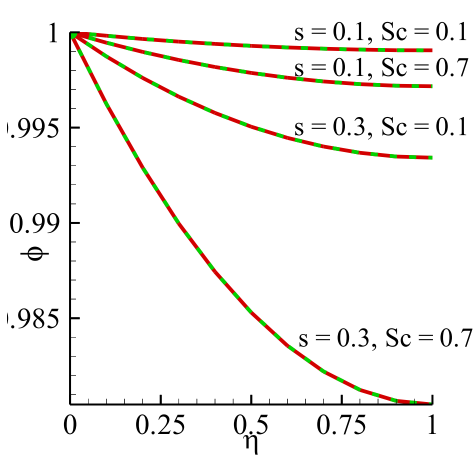


**Figure 4.** Concentration profiles for various values of s and Sc while $\gamma$ =0.1, r=0.1, Pr=0.1.

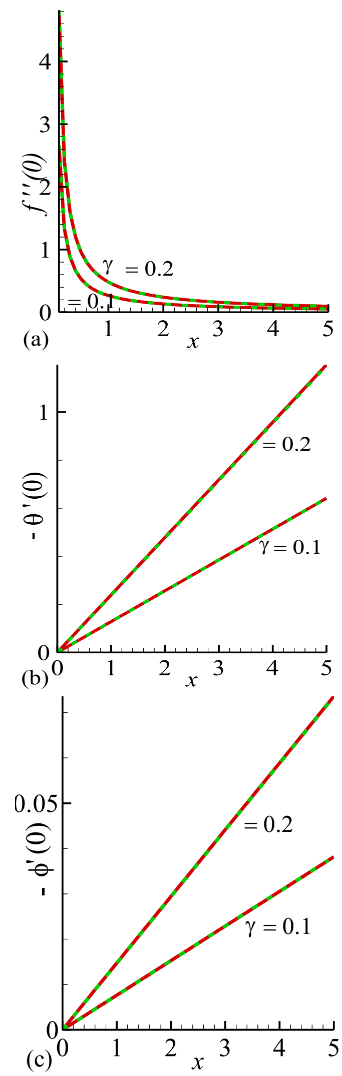


**Figure 5.** (a), Skin-friction; (b), heat transfer rate; (c), mass transfer rate for different values of $\gamma$ while Sc=0.2, s=0.2, Pr=0.7 and r=1.0.

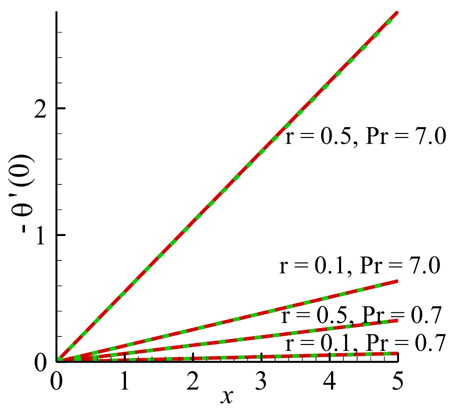


**Figure 6.** Heat transfer rate for various values of r and Pr while $\gamma$ =0.1, s=0.1, Sc=0.1.

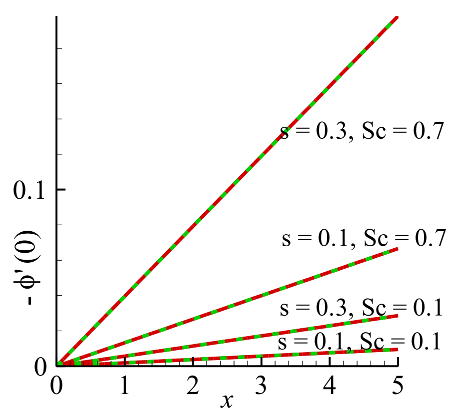


**Figure 7.** Mass transfer rate for various values of s and Sc while $\gamma$ =0.1, r=0.1, Pr=0.1.

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